Convolutional perfectly matched layer for isotropic and anisotropic acoustic wave equations

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Summary

A new convolutional Perfectly Matched Layer (CPML) absorbing boundary condition is derived for isotropic and anisotropic acoustic wave equations. The CPML equations are expressed as the regular wave equations with additional correction terms, making the integration of the new condition into the existing seismic software relatively easy. Finally, effectiveness of the CPML is demonstrated on the example of a homogenous TTI model. For seismic imaging applications, it should be possible to use a 3 layer thick CPML. With as few as 5 layers, reflection coefficients can be reduced to 1%.

Introduction

The Perfectly Matched Layer (PML), first introduced by Berenger (1994), has proven to be one of the most robust and efficient absorbing boundary conditions for termination of finite computational domains. Since its introduction, many alternative formulations of PML have been proposed to simplify its implementation and improve effectiveness. One of the most widely used formulations is the convolutional PML (CPML), proposed by Roden and Gedney (2000). This is a stretched coordinate PML formulation, using an additional parameter (\(\alpha\)) to control late time low-frequency reflection and improve absorption of grazing waves.

The PML was initially introduced for simulations of electromagnetic problems, and subsequently adopted in other areas. In seismic modeling, different versions of PML have been successfully applied to both acoustic (e.g. Qi and Geers (1998), Hu et al. (2007)) and elastic problems (e.g. Hastings et al. (1996), Komatitsch and Martin (2007)). Most of these applications involve systems of first-order differential equations for which derivation of the PML boundary condition is relatively straightforward. A few derivations have been proposed for systems of second-order differential equations. Some examples include: Komatitsch and Tromp (2003) who derived a split-field PML for elastic wave equations, and McGarry and Moghaddam (2009) who derived an NPML for acoustic wave equations written in second-order differential form.

In this paper, we derive a CPML for isotropic and anisotropic acoustic wave equations. The proposed formulation is both highly efficient and easy to implement in existing acoustic wave propagation software. A major advantage of the CPML over the classical formulation is an additional degree of freedom which allows better absorption of waves at grazing angles. This is especially important if a source is located close to the edge of the mesh or receivers are located at very large offset (e.g. Komatitsch and Martin (2007)). Furthermore, the classical PML can suffer from late-time low frequency reflection which can cause instability of the algorithm. As shown by Bécache et al. (2004), this problem is eliminated in CPML. Efficiency of the new CPML boundary condition is demonstrated on the example of tilted transverse isotropic (TTI) acoustic modeling.

Theory

The CPML equations are obtained by introducing a complex coordinate stretching parameter to the acoustic field equations in the frequency-domain and applying the recursive convolution algorithm to convert the equations to the time-domain. The stretching parameter in the \(t\)-direction (\(t \in \{x,y,z\}\)) is given in the frequency-domain by

\[
s_t = 1 + \frac{\sigma_t}{\delta_t + j\omega}\quad (1)
\]

where \(\sigma_t\) is a damping factor which controls absorption of the acoustic wave, \(\omega\) is angular frequency, \(\delta_t^2 = -1\), and \(\delta_t\) is a positive, real parameter causing pole shifting off the origin into the upper-half complex plane. The latter is specific only to CPML and improves absorption of low-frequency components. The stretching parameter defined by (1) is referred to as the complex frequency shifted (CFS) tensor coefficient.

In the time-domain, the partial derivative in the stretched coordinate space is given by

\[
\frac{\partial}{\partial t} = \mathcal{F}^{-1}\left\{ \frac{1}{s_t} \right\} * \frac{\partial}{\partial t} \quad (2)
\]

where \(\mathcal{F}^{-1}\) denotes the inverse Fourier transform (IFT), and tilde denotes the stretched coordinate. The IFT of the inverse CFS tensor coefficient can be written as

\[
\mathcal{F}^{-1}\left\{ \frac{1}{s_t} \right\} = \delta(t) - \sigma_t e^{-(\delta_t + j\omega)t}u(t)\quad (3)
\]

where \(\delta(t)\) is the Dirac delta function and \(u(t)\) is the step function. The convolution in (2) can be solved efficiently using the recursive convolution method (Roden and Gedney (2000)) as follows

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \psi_t \quad (4)
\]

where \(\psi_t\) is an auxiliary variable whose time-evolution is given by...
In the above equation, \( n \) denotes the current time level for calculations and parameters \( a_i \) and \( b_i \) are given by

\[
a_i = e^{-(\sigma_i + \alpha_i)\Delta t}, \quad b_i = \frac{\sigma_i}{\sigma_i + \alpha_i} (a_i - 1) \tag{6}
\]

Application to Isotropic Acoustic Wave Equation

The acoustic scalar wave equation for isotropic media can be written as

\[
\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \tag{7}
\]

where \( c \) is the acoustic velocity and \( P \) is the scalar field. The above equation contains second partial derivatives of the scalar field. For easier CPML implementation, the second spatial derivatives should be expressed in terms of the corresponding first derivatives as

\[
\frac{\partial^2 P}{\partial t^2} = \frac{\partial P}{\partial t} - \frac{\partial P}{\partial x} \frac{\partial x}{\partial t} \quad \frac{\partial^2 P}{\partial x^2} = \frac{\partial^2 P}{\partial x^2} - \frac{\partial x}{\partial t} \frac{\partial P}{\partial x} - \frac{\partial^2 P}{\partial x \partial y} \frac{\partial y}{\partial t} \tag{8}
\]

Using the above formulation, the acoustic wave equation in the stretched coordinate space can be written as

\[
\frac{1}{c^2} \frac{\partial^2 p_i}{\partial t^2} = \frac{\partial^2 p_i}{\partial x^2} + \frac{\partial^2 p_i}{\partial y^2} + \frac{\partial^2 p_i}{\partial z^2} \tag{9}
\]

where \( p_i, (i \in \{x, y, z\}) \) is defined as

\[
p_i = \frac{\partial P}{\partial t} \tag{10}
\]

Applying (4) to the above equation, \( p_i \) can be expressed as

\[
p_i = \frac{\partial P}{\partial t} + \psi_i \tag{11}
\]

where \( \psi_i \) is the auxiliary variable with time-evolution given by (5). Applying (4) and (11) to (9), the isotropic acoustic equation in the stretched coordinate space becomes

\[
\frac{1}{c^2} \frac{\partial^2 p_i}{\partial t^2} = \frac{\partial^2 p_i}{\partial x^2} + \frac{\partial^2 p_i}{\partial y^2} + \frac{\partial^2 p_i}{\partial z^2} + \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} + \frac{\partial^2 \psi_i}{\partial z^2} + \xi_x + \xi_y + \xi_z \tag{12}
\]

where \( \xi_i, (i \in \{x, y, z\}) \) are auxiliary variables with time-evolution given by

\[
\xi_i^n = a_i \xi_i^{n-1} + b_i \left( \frac{\partial^2 \psi_i}{\partial t^2} \right)^n + \left( \frac{\partial \psi_i}{\partial t} \right)^n \tag{13}
\]

Parameters \( a_i \) and \( b_i \) are given by (6).

Equations (12) and (13) together with (5) make up a full set of equations to be solved within the CPML. The equations are presented in a form similar to the regular scalar acoustic equation (7) with the correction terms. Within the inner computational domain, outside the PML regions, only the regular scalar equation (7) needs to be solved.

The correction terms in (12) contain a total of six auxiliary variables. However, one has to note that the auxiliary variables \( \psi_i \) and \( \xi_i \) are nonzero only inside the PML regions perpendicular to the \( i \)-direction. In the PML region perpendicular to the \( x \)-direction, away from the corner areas, the CPML equation (12) simplifies to

\[
\frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} + \frac{\partial \psi_x}{\partial x} + \xi_x \tag{14}
\]

The full equation (12) needs to be considered only in 3-way corners.

Application to VTI Acoustic Wave Equations

The basic (pseudo-)acoustic wave equations we use for vertical transverse isotropic (VTI) media can be expressed as (McGarry and Moghaddam (2009))

\[
\frac{\partial^2 p_x}{\partial t^2} = c_x \frac{\partial^2 p_x}{\partial x^2} + c_y \frac{\partial^2 p_y}{\partial y^2} + c_z \frac{\partial^2 p_z}{\partial z^2} \tag{14}
\]

where \( P \) and \( R \) are the pseudo-acoustic wavefields, and \( c_i, d_i \) \((i \in \{x, y, z\}) \) are velocity coefficients given by

\[
c_x = c_y = \nu \nu (1 + 2 \varepsilon) ; \quad c_z = \nu \nu \nu
\]

\[
d_x = d_y = \nu (1 + 2 \delta) ; \quad c_z = \nu \nu
\]

\( \nu \nu \) is the vertical velocity and \( \varepsilon, \delta \) are the usual Thomsen anisotropy parameters.

The CPML equations for VTI media are derived following a similar procedure used for isotropic media. Using (8) and introducing stretched coordinates, the CPML equations can be written as

\[
\frac{\partial^2 p_x}{\partial t^2} = c_x \frac{\partial^2 p_x}{\partial x^2} + c_y \frac{\partial^2 p_y}{\partial y^2} + c_z \frac{\partial^2 p_z}{\partial z^2} \tag{15}
\]

\[
\frac{\partial^2 R}{\partial t^2} = d_x \frac{\partial^2 p_x}{\partial x^2} + d_y \frac{\partial^2 p_y}{\partial y^2} + d_z \frac{\partial^2 p_z}{\partial z^2}
\]

where \( p_i, (i \in \{x, y\}) \) is given by (10) and \( R \) is given by
Using (4), the expressions for \( p_i \) and \( r_z \) can be reformulated as

\[
p_i = \frac{\partial P}{\partial i} + \psi_{p,i}; \quad r_z = \frac{\partial R}{\partial z} + \psi_{r,x}
\]  

(17)

According to (5), the time-evolution of the auxiliary variables \( \psi_{p,i} \) and \( \psi_{r,x} \) can be expressed as

\[
\psi^n_{p,i} = a_i \psi^{n-1}_{p,i} + b_i \left( \frac{\partial P}{\partial i} \right)^n
\]

(18)

and

\[
\psi^n_{r,x} = a_i \psi^{n-1}_{r,x} + b_i \left( \frac{\partial R}{\partial z} \right)^n
\]

where \( a_i \) and \( b_i \) are given by (6).

Applying (17) to (15), and using (4), the CPML equations for VTI media are obtained as

\[
\frac{\partial^2 P}{\partial t^2} = c_x \frac{\partial^2 P}{\partial x^2} + c_y \frac{\partial^2 P}{\partial y^2} + c_z \frac{\partial^2 P}{\partial z^2} + c_{xy} \frac{\partial^2 P}{\partial x \partial y} + c_{xz} \frac{\partial^2 P}{\partial x \partial z} + c_{yz} \frac{\partial^2 P}{\partial y \partial z}
\]

\[
+\frac{\partial^2 R}{\partial x^2} + \psi_{p,x} \left( \frac{\partial \psi_{p,y}}{\partial y} + \psi_{r,y} \right)
\]

\[
+\frac{\partial^2 R}{\partial y^2} + \psi_{r,x} \left( \frac{\partial \psi_{r,y}}{\partial y} + \psi_{r,y} \right)
\]

\[
\frac{\partial^2 R}{\partial t^2} = d_x \frac{\partial^2 P}{\partial x^2} + d_y \frac{\partial^2 P}{\partial y^2} + d_z \frac{\partial^2 P}{\partial z^2} + d_{xy} \frac{\partial^2 P}{\partial x \partial y} + d_{xz} \frac{\partial^2 P}{\partial x \partial z} + d_{yz} \frac{\partial^2 P}{\partial y \partial z}
\]

\[
+\frac{\partial^2 R}{\partial x^2} + \psi_{r,x} \left( \frac{\partial \psi_{r,y}}{\partial y} + \psi_{r,y} \right)
\]

\[
+\frac{\partial^2 R}{\partial y^2} + \psi_{p,x} \left( \frac{\partial \psi_{p,y}}{\partial y} + \psi_{p,y} \right)
\]

(19)

The auxiliary variables \( \psi_{p,i} \) and \( \psi_{r,x} \) are updated in time as

\[
\psi^n_{p,i} = a_i \psi^{n-1}_{p,i} + b_i \left( \frac{\partial^2 P}{\partial i} \right)^n
\]

(20)

\[
\psi^n_{r,x} = a_i \psi^{n-1}_{r,x} + b_i \left( \frac{\partial^2 R}{\partial z} \right)^n
\]

Equations (18) - (20) constitute a full set of equations to be solved inside the PML regions. Within the inner computational domain, outside the PML regions, only the basic VTI equations (14) need to be solved.

The VTI CPML equations are expressed as the basic VTI equations (14) with correction terms comprising six auxiliary variables. However, the auxiliary variables \( \psi_{p(r)i} \) and \( \psi_{r(r,i)} \) (\( i \in \{x, y, z\} \)) are nonzero only in the PML regions perpendicular to the \( i \)-direction. Therefore, in most of the CPML, only two auxiliary variables are used. Four and six auxiliary variables are used only in 2-way and 3-way corner regions, respectively.

### Application to TTI Acoustic Wave Equations

The TTI acoustic wave equations are obtained by rotating the VTI equations (14) according to the following matrix

\[
R = \begin{bmatrix}
\cos \theta & \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0 & \sin \theta \\
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta 
\end{bmatrix}
\]

(21)

where \( \theta \) and \( \phi \) are the dip and azimuth angles, respectively. After rotation, the TTI equations become (for compactness only \( P \) wavefield equation is given)

\[
\frac{\partial^2 P}{\partial t^2} = c_x \frac{\partial^2 P}{\partial x^2} + c_y \frac{\partial^2 P}{\partial y^2} + c_z \frac{\partial^2 P}{\partial z^2} + c_{xy} \frac{\partial^2 P}{\partial x \partial y} + c_{xz} \frac{\partial^2 P}{\partial x \partial z} + c_{yz} \frac{\partial^2 P}{\partial y \partial z}
\]

\[
+\frac{\partial^2 R}{\partial x^2} + \psi_{p,x} \left( \frac{\partial \psi_{p,y}}{\partial y} + \psi_{r,y} \right)
\]

\[
+\frac{\partial^2 R}{\partial y^2} + \psi_{r,x} \left( \frac{\partial \psi_{r,y}}{\partial y} + \psi_{r,y} \right)
\]

\[
\frac{\partial^2 R}{\partial t^2} = d_x \frac{\partial^2 P}{\partial x^2} + d_y \frac{\partial^2 P}{\partial y^2} + d_z \frac{\partial^2 P}{\partial z^2} + d_{xy} \frac{\partial^2 P}{\partial x \partial y} + d_{xz} \frac{\partial^2 P}{\partial x \partial z} + d_{yz} \frac{\partial^2 P}{\partial y \partial z}
\]

\[
+\frac{\partial^2 R}{\partial x^2} + \psi_{r,x} \left( \frac{\partial \psi_{r,y}}{\partial y} + \psi_{r,y} \right)
\]

\[
+\frac{\partial^2 R}{\partial y^2} + \psi_{p,x} \left( \frac{\partial \psi_{p,y}}{\partial y} + \psi_{p,y} \right)
\]

(22)

The expression for the auxiliary wavefield \( R \) has a similar form. Coefficients \( c_i, c_{ij}, d_i, \) and \( d_{ij} \) are functions of vertical velocity, dip and azimuth angles, and the Thomsen anisotropy parameters.

The CPML equations for TTI media can be derived from (22) by following a similar procedure to that used for VTI media. An additional complexity is the existence of the cross-derivative terms. They can be expressed in terms of the first derivatives as

\[
\frac{\partial^2 P}{\partial i^2} = \frac{\partial}{\partial i} \left( \frac{\partial P}{\partial i} \right)
\]

(23)

where \( i, j \in \{x, y, z\} \).

By introducing the stretched coordinates to (22) and using (8) and (23), one can derive a set of CPML equations for TTI media. These equations can be expressed in a form similar to that of the basic TTI wave equations (22) plus the correction terms. The correction terms contain nine auxiliary variables per wavefield. However, all nine variables are used only in 3-way corner regions. In the PML areas away from the corner regions, only three variables per wavefield are used.
CPML for acoustic wave equations

Numerical Example

The effectiveness of the new CPML boundary condition is demonstrated on a 2-D TTI model with a constant vertical velocity of 2000 m/s. The anisotropic Thomsen parameters $\varepsilon$ and $\delta$ are 0.2 and 0.05, respectively. The assumed dip angle is $45^\circ$. The model dimensions are 2 km x 2 km and the cell size is 8 m. The simulation is excited by a Ricker wavelet with a peak frequency ($f_0$) of 30 Hz. The excitation point is located at the center of the computational domain. Wave propagation equations are solved using a 10th order in space and 2nd order in time finite difference scheme.

Figure 1 shows the snapshots of $P$ wave time evolution created in our simulation. An 8-layer CPML boundary is used for wave absorption. In this case the reflection coefficient in the frequency range of interest ($0.1f_0 - 2f_0$) is below 0.2%. For 5 and 3 layer CPML boundaries the obtained maximum reflection coefficients are 1% and 5%, respectively.

Figure 2 shows decay of total energy in the grid. Two steps can be observed in the energy diagram. The first step corresponds to both main and shear waves being present in the grid. After 0.8 s, the main $P$ and $R$ waves are fully absorbed by the CPML and only the shear waves remain. This corresponds to the second step in the graph. Finally, after approximately 1.2 s, the shear waves are absorbed by the CPML.

Conclusions

We have derived a CPML boundary condition for acoustic isotropic and anisotropic (VTI and TTI) wave equations. The CPML equations are expressed in a form similar to the original wave equations with added correction terms. Therefore, implementation of this boundary condition into existing acoustic wave propagation software is straightforward. The efficiency of the new absorbing boundary is demonstrated on an example 2-D TTI constant velocity model. It is shown that, besides the main $P$ and $R$ waves, the CPML readily absorbs the unwanted pseudo-shear waves as well. A 3-layer CPML should be sufficient for seismic imaging applications. A 5-layer CPML reduces reflections to around 1%.

Figure 1 Propagation of $P$ wave in a homogenous 2D TTI medium. From top to bottom, the images correspond to 0.48s and 0.67s.

Figure 2 Decay of field energy with time. CPML is used as the absorbing boundary condition. The graph clearly indicates that even the unwanted shear waves are absorbed by the CPML.